INDIAN STATISTICAL INSTITUTE
Probability Theory II: B. Math (Hons.) I
Semester II, Academic Year 2017-18
Semestral Examination

Teacher: Parthanil Roy

Date: 04/05/2018 Total Marks: 50 Duration: 10:00 am - 1:00 pm

Note:

- **Please write your name on your answer booklet.**
- **Show all your works and write explanations when needed.**
- **You may use any fact stated and/or proved in the class but do not forget to quote the appropriate result.**

1. A continuous random vector (X, Y) has a joint probability density function given by

$$f_{X,Y}(x,y) = \begin{cases} c & \text{if } x > 0, y > 0, x + y < 1, \\ 4/3 & \text{if } x < 1, y < 1, x + y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find c. [2]

   (b) Compute the marginal probability density function of X. [4]

   (c) Compute the conditional probability density function of Y given X. [4]

   (d) Calculate $E(Y|X)$ and $Var(Y|X)$. $[4 + 4 = 8]$

2. Let X be a random variable and $\{X_n\}_{n\geq 1}$ be a sequence of random variables such that $X_n \xrightarrow{d} X$ as $n \to \infty$. Show that $X_n^2 \xrightarrow{d} X^2$ as $n \to \infty$. [7]

3. Let X_1, X_2, X_3, X_4 be i.i.d. standard normal random variables. For $k = 1, 2, 3$, define

$$Y_k = \frac{1}{\sqrt{k(k+1)}} \left(\sum_1^k X_i - kX_{k+1} \right).$$

   Show that Y_1, Y_2, Y_3 are also i.i.d. standard normal random variables. [10]

4. Suppose X follows a double-exponential distribution with a probability density function

$$f_X(x) = \frac{1}{2} e^{-|x|}, \quad x \in \mathbb{R}.$$

   Compute the characteristic function of X. [5]

[P. T. O]

5. Suppose $Y_1, Y_2, \ldots, Y_n$ is a random sample from a standard Cauchy distribution with common probability density function

$$g(y) = \frac{1}{\pi(1+y^2)}, \quad y \in \mathbb{R}.$$

Compute a probability density function of $\frac{1}{n}\sum_{i=1}^{n} Y_i$. Justify all your steps. [10]

Wish you all the best